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Dynamical ultrametricity in the critical trap model

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Abstract

We show that the trap model at its critical temperature presents dynamical ultrametricity in the sense of Cugliandolo and Kurchan (Cugliandolo L F and Kurchan J 1994 *J. Phys. A: Math. Gen.* **27** 5749). We use the explicit analytical solution of this model to discuss several issues that arise in the context of mean-field glassy dynamics, such as the scaling form of the correlation function, and the finite time (or finite forcing) corrections to ultrametricity, that are found to decay only logarithmically with the associated timescale, as well as the fluctuation–dissipation ratio. We also argue that in the multi-level trap model, the short-time dynamics is dominated by the level which is at its critical temperature, so that dynamical ultrametricity should hold in the whole glassy temperature range. We revisit some experimental data on spin glasses in the light of these results.

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1. Introduction

Notable theoretical progress in our understanding of the ubiquitous *ageing* phenomena in glassy systems was made possible by the recognition that the discussion of correlation and response functions requires *two* times: the waiting time t_w and the total elapsed time $t_w + t$. This appears very clearly in the framework of mean-field spin-glass models, domain-growth (coarsening) models, or more phenomenological trap models [2][1]. Although the basic phenomenology of all these models is rather similar, the underlying physical picture is completely different. For example, ageing in domain growth models is associated with the growth of a coherence length. In mean-field or trap models, space is absent and ageing is related to the structure of the energy landscape, but here again the intuition is completely different. In mean-field models, the system never reaches the bottom of an energy valley and there are no activated processes involved in the dynamics. Rather, the dynamics slows

[1] Since here we repeatedly compare trap models and mean-field spin-glass models, we will refer to the latter class simply as mean-field models, even though the fully connected trap model that we consider hereafter might also deserve such a denomination.

down because saddles with less and less 'descending' directions are visited as the system ages. Conversely, in the trap model, activation is the basic ingredient of the model, and ageing is associated with the fact that deeper and deeper valleys are reached as the system evolves. Dynamics in the latter case is fundamentally *intermittent*: either nothing moves, or there is a jump between two traps. This must be contrasted with mean-field dynamics which is continuous in time. However, mean-field and trap dynamics can be shown to correspond to two successive time regimes in a particular class of models [3].

In spite of these important differences, many predictions are common to the latter two scenarios, such as:

- A short time singularity of the response function in the ageing regime, which leads to an (ageing) low-frequency noise.
- Non-trivial violations of the fluctuation dissipation theorem (FDT), first pointed out within mean-field models, but that also exists in trap models.
- The possibility of rejuvenation and memory, which involves the existence of different degrees of freedom with different timescales.

Furthermore, a certain class of mean-field models (that includes the Sherrington–Kirkpatrick (SK) model) has been conjectured to possess ultrametric dynamical properties, that very precisely reflect and encode the ultrametric nature of the static solution. This dynamical ultrametricity is associated with an infinite number of timescales (which diverges with the age of the system) in the following sense: if $C(t_2, t_1)$ is the correlation function between times $t_1 < t_2$, then in the limit of large times:

$$C(t_3, t_1) = \min(C(t_2, t_1), C(t_3, t_2)) \qquad \forall t_2 \in [t_1, t_3]. \tag{1}$$

This means that either t_2 is close enough to t_3, and then no further dynamics takes place between t_2 and t_3, or t_2 is close enough to t_1 but then the age of the system hardly changes between t_1 and t_2. This property of the correlation function has been shown to hold for the ageing solution of the dynamical (mode coupling) equations describing the dynamics of 'continuous' spin glasses [1]. Furthermore this property is invariant under reparametrizations of time, where $t \to h(t)$ with an arbitrary monotonic function $h(t)$. Testing whether or not dynamical ultrametricity also holds in realistic disordered systems is made difficult because this property should only hold in the limit of asymptotically large times, and corrections on finite times are expected. How large are these corrections?

In this paper, we show that exact ultrametricity holds *at* the critical point of the single-level trap model (or random energy model). We give the explicit form of the correlation function and discuss finite time (or finite forcing) corrections. Note that in this single-level trap model, the dynamics is ultrametric although the statics is not. The issue of finite time FDT plot is also addressed. We discuss multi-level extensions of the trap model and argue that dynamical ultrametricity should be generic at 'short times', i.e. at the beginning of the ageing region. We show the thermoremanent magnetization (TRM) data that support this idea. The relation with $1/f$ noise, already discussed in this context [7], is recalled.

2. The model

The trap model, introduced in the context of ageing in [4, 5] and further developed in [6], is one of the simplest soluble models exhibiting a dynamical glass transition. In this model, one considers a particle which is trapped in low-energy states i of depth $E_i > 0$, where the E_i are random variables distributed according to $\rho(E) = \frac{1}{T_c}\mathrm{e}^{-E/T_c}$. The dynamics is chosen to be activated: each particle stays in trap i for an exponential random time, on average equal to

$\tau_0\, e^{E_i/T}$. The quantity τ_0 is a microscopic timescale which we shall take as the time unit in the following. When the particle leaves the trap, it randomly chooses a new one among all the others. As a consequence, at a high temperature ($T > T_c$), the particle spends most of its time in the small traps, because the number of these traps (the entropic factor) dominates the Boltzmann factor and the system equilibrates. In contrast, for $T < T_c$, the Boltzmann factor is dominant and the particle explores deeper and deeper traps, so that the system never equilibrates (in the limit of infinite number of traps). In this regime, the dynamics ages: correlation and response functions are no longer time translation invariant, but depend on both the waiting time t_w and the total time $t_w + t$.

3. Correlation function

Correlation functions are useful tools for characterizing the dynamics and comparing several models. The simplest (but nevertheless non-trivial) correlation in the trap model is $\Pi(t_w + t, t_w)$, defined as the probability of remaining in the same trap during the time interval $[t_w, t_w + t]$. As we are considering the infinite-dimensional (or fully connected) model, this probability is also equal to the probability $P(t_w + t, t_w)$ to be in the same trap at $t_w + t$ as at t_w, since the probability of going back to the same trap vanishes in the limit of infinite number of traps. (This is not true in finite dimensions. For instance, in one dimension, $\Pi(t_w + t, t_w)$ and $P(t_w + t, t_w)$ scale in a different way with t_w.)

The calculation of the correlation function $\Pi(t_w + t, t_w)$ in the critical case $T = T_c$ is given in the appendix. In the limit where $\log t_w$ and t are both large, we have up to first order in a $1/\log t_w$ expansion

$$\Pi(t_w + t, t_w) \simeq \frac{\log\left(1 + \frac{t_w}{t}\right)}{\log t_w}. \tag{2}$$

This relation shows that $\Pi(t_w + t, t_w)$ is not a function of $\frac{t}{t_w}$, at variance with the results that hold in the whole low-temperature phase $T < T_c$ [7]. In contrast, taking $t \sim a t_w^{\alpha}$, we obtain, in the limit t_w going to $+\infty$,

$$\Pi(t_w + t, t_w) = \mathcal{C}(\alpha) \tag{3}$$

with $\mathcal{C}(\alpha)$ given by

$$\mathcal{C}(\alpha) = 1 - \alpha \qquad (\alpha < 1) \tag{4}$$

$$= 0 \qquad (\alpha \geqslant 1). \tag{5}$$

Note that $\mathcal{C}(\alpha)$ is a monotonic decreasing function of α.

Interestingly, a similar scaling has also been found recently in the voter model [8]. To be more specific, it was shown that the correlation function $C_{\mathrm{vot}}(t_w + t, t_w)$ defined by

$$C_{\mathrm{vot}}(t_w + t, t_w) = \langle S(t_w) S(t_w + t)\rangle \tag{6}$$

is given by

$$C_{\mathrm{vot}}(t_w + t, t_w) = \frac{\log\left(1 + \frac{2t_w}{t}\right)}{\log t_w} \tag{7}$$

to first order in $1/\log(t_w)$, which reduces in the infinite time limit to $C_{\mathrm{vot}}(t_w + t, t_w) = \mathcal{C}(\alpha)$, with the same $\mathcal{C}(\alpha)$ as above.

An important remark is that the correlation function $\Pi(t_w + t, t_w)$ is a function of $\alpha = \log t / \log t_w$ that cannot be written as $h(t_w + t)/h(t_w)$. The latter ratio naturally appears (with an unknown function h) in the ageing part of the solution of the dynamical equation corresponding to one-step replica symmetry breaking (RSB) mean-field spin-glass models. However, for full RSB models where dynamical ultrametricity is indeed expected, the correlation function is given by an infinite sum of contributions coming from different time sectors [1, 2]:

$$C(t_w + t, t_w) = \sum_i \mathcal{C}_i \left(\frac{h_i(t_w + t)}{h_i(t_w)} \right) \tag{8}$$

where h_i are unknown (monotonic) functions defining the ith timescale, and the $\mathcal{C}_i$ monotonically decay to zero for large arguments.

A useful (but up to now theoretically unjustified) form for $h_i(t)$, that allows one to give some flesh to the above formula, is [2]:

$$h_i(u) = \exp\left[\frac{u^{1-\mu_i}}{1-\mu_i} \right] \qquad 0 \leqslant \mu_i \leqslant 1. \tag{9}$$

It is easy to see that for this choice of h_i, the timescale on which the ratio $h_i(t_w + t)/h_i(t_w)$ varies significantly is precisely $t_w^{\mu_i}$. The choice $\mu = 0$ therefore corresponds to stationary dynamics, whereas $\mu = 1$ gives full ageing. Now, we take $t = t_w^{\alpha}$ (with $0 < \alpha < 1$) in equation (8) and the limit $t_w \to \infty$. All the sectors such that $\mu_i < \alpha$ have relaxed to zero, whereas the sectors corresponding to $\mu_i > \alpha$ have not decayed at all. Introducing a continuum of different values of μ, we find that the correlation function is given by

$$C\left(t_w + t_w^{\alpha}, t_w\right) = \int_{\alpha}^{1} \mathrm{d}\mu\, \rho(\mu) \mathcal{C}_{\mu}(1) \tag{10}$$

where $\rho(\mu)$ is the 'density' of time sectors of order t_w^{μ} and $\mathcal{C}_{\mu}(1)$ is the initial value of the correlation function in this sector. From this result, one sees that $C(t_w + t, t_w)$ indeed becomes a function of $\alpha = \log t / \log t_w$ in the long time limit. Therefore, interestingly, the superposition of an infinite number of *subageing* contributions defined by (9) naturally leads to a correlation function that depends on $\log t / \log t_w$, for which the dynamical ultrametricity property is explicit. The critical trap behaviour corresponds to a uniform contribution of all time sectors, i.e. $\rho(\mu)\mathcal{C}_{\mu}(1) = 1, \forall \mu$.

4. Dynamical ultrametricity

As recalled in the introduction, Cugliandolo and Kurchan have defined dynamical ultrametricity for the correlation function C if the following property is true: on taking three times $t_1 < t_2 < t_3$, in the limit of large times,

$$C(t_3, t_1) = \min\left(C(t_2, t_1), C(t_3, t_2)\right) \qquad \forall t_2 \in [t_1, t_3]. \tag{11}$$

Let us now show that $\Pi(t_w + t, t_w)$ at the critical temperature is ultrametric in the sense defined above. It will be useful to introduce the following notation:

$$\Pi(t_2, t_1) = C_1 \qquad \Pi(t_3, t_2) = C_2 \qquad \Pi(t_3, t_1) = C_3. \tag{12}$$

Since correlation functions are monotonic, the inequality $C_3 \leqslant \min(C_1, C_2)$ holds in general. We simply have to check that (at least) one of the two correlations C_1 and C_2 is equal to C_3. In order to take the infinite time limit, we need to specify how t_2 and t_3 scale with t_1. A natural parametrization, that leads to non-trivial values of the C_i, is the following:

$$t_3 - t_1 \sim a t_1^{\alpha} \qquad t_2 - t_1 \sim b t_1^{\beta}. \tag{13}$$

We can now look at various cases:

- If $\beta < \alpha$, then one has, for large t_1:

$$t_3 - t_2 \sim a t_1^{\alpha} - b t_1^{\beta} \sim a t_1^{\alpha} \tag{14}$$

so that $C_1 = \mathcal{C}(\beta)$ and $C_2 = C_3 = \mathcal{C}(\alpha) < \mathcal{C}(\beta)$.
- Now assuming $\beta = \alpha$, we get $C_1 = C_3 = \mathcal{C}(\alpha)$, as well as $C_2 \geqslant C_3$.

As a result, we have shown that the relation $C_3 = \min(C_1, C_2)$ always holds, which implies that dynamical ultrametricity is satisfied in this model.

The appearance of dynamical ultrametricity can be considered as a signature of the existence of many timescales involved in the dynamics. This is indeed the case in the present model, even though there is no static ultrametricity to account for this hierarchy of timescales. This property in fact arises naturally from the exact balance, at the critical point, of the Boltzmann weight $e^{E/T}$ and of the entropic factor $\rho(E)$: the probability for the particle to have a given energy (or equivalently, a given trapping time, in logarithmic scale) is essentially uniform in the interval $[0, T_c \log t_w]$. In other words, dynamical ultrametricity here is a consequence of the critical scale invariance.

5. Finite time analysis

An interesting analysis was introduced by Cugliandolo and Kurchan in the context of the dynamical analysis of the SK model [1]. These authors made the assumption that, in the limit of large times, there exists a certain function $f(x, y)$, not necessarily smooth such that $C_3 = f(C_1, C_2)$. We have shown in the previous section that such a function indeed exists in the present case and is given by $f(x, y) = \min(x, y)$.

A useful representation, proposed in [1], is to plot the curves of constant $C_3 = f(C_1, C_2)$ in the (C_1, C_2)-plane, which reduces for our case to two straight lines ($C_1 = C_3$ and $C_2 = C_3$) at right angles. For finite times, the function f has to include a timescale as the third argument, so that $C_3 = F(C_1, C_2; t_1)$, where $f(x, y) = \lim_{t_1\to\infty} F(x, y; t_1)$. The (C_1, C_2)-plane representation is then a good way to visualize the convergence towards the asymptotic function $f(C_1, C_2)$. It has been used with numerical data to test if dynamic ultrametricity holds in realistic systems, with rather inconclusive results [1, 9], except in a few cases, such as in a recent study of the four-dimensional Edwards–Anderson model [10].

Let us apply this procedure to the critical trap model. In order to deal with finite time expressions, we shall go back to equation (2), restated as

$$C(t_j, t_i) = \frac{\log\left(1 + \frac{t_i}{t_j - t_i}\right)}{\log t_i}. \tag{15}$$

Note that all the finite time results reported in this paper should be understood as first-order terms in a $1/\log t_1$ expansion. Inverting these relations to express t_2 and t_3 as functions of t_1, C_1 and C_3, we can write an explicit expression for $F(C_1, C_2; t_1)$:

$$\begin{aligned} C_3 &= F(C_1, C_2; t_1) \\ &= -\frac{1}{\log t_1} \log\left(t_1^{-C_1} + t_1^{-C_2}\left(1 - t_1^{-C_1}\right)^{1+C_2}\right). \end{aligned} \tag{16}$$

In order to plot the constant C_3 curves, it will be useful to also express C_2 as a certain function $G(C_1, C_3; t_1)$:

$$C_2 = G(C_1, C_3; t_1) = \frac{\log\left(1 - t_1^{-C_1}\right) - \log\left(t_1^{-C_3} - t_1^{-C_1}\right)}{\log t_1 - \log\left(1 - t_1^{-C_1}\right)}. \tag{17}$$

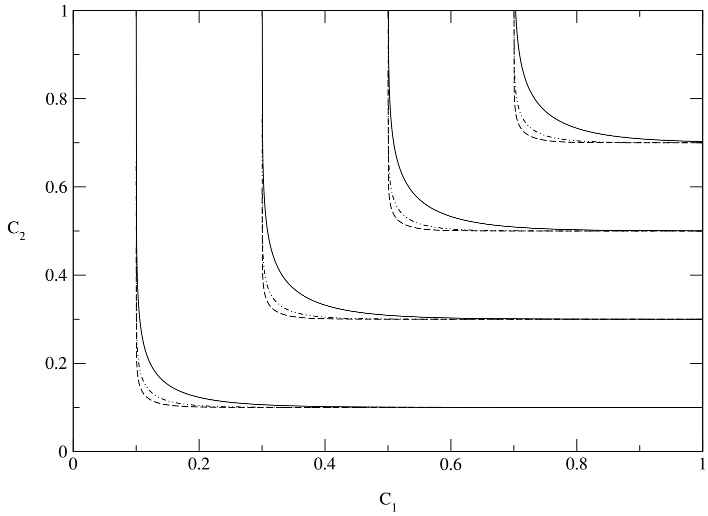

Figure 1. Plot of constant $C_3 = F(C_1, C_2; t_1)$ in the (C_1, C_2)-plane. From left to right, the curves correspond to $C_3 = 0.1, 0.3, 0.5, 0.7$. Each set of three curves shows the convergence with t_1 towards the asymptotic function $f(C_1, C_2) = \min(C_1, C_2)$: $t_1 = 10^5$ (full line), 10^{10} (dot–dashed line), 10^{15} (dashed line).

The resulting plots in the (C_1, C_2)-plane are displayed in figure 1, for $C_3 = 0.1, 0.3, 0.5, 0.7$ and for three different times $\left(t_1 = 10^5, 10^{10} \text{ and } 10^{15}\right)$. Note that the convergence is very slow close to the infinite time singularity.

Cugliandolo and Kurchan also introduced in [1] the notion of correlation timescales through the representation of $f(C, C)$ versus C. As already mentioned before, $f(C, C) \leqslant C$ in general. There may exist some special 'fixed' points C^* such that $f(C^*, C^*) = C^*$. Each of these fixed points has been shown to be associated with a correlation timescale. If dynamical ultrametricity holds in a particular time sector, all C belonging to a certain interval $[C', C'']$ are fixed points. In our case, ultrametricity holds over the full correlation interval [0, 1]. But in our model, we can go beyond the infinite time analysis and quantify the convergence of $F(C, C; t_1)$ with t_1 towards the asymptotic function $f(C, C) = C$. We find

$$F(C, C; t_1) = C - \frac{\log\left(1 + \left(1 - t_1^{-C}\right)^{1+C}\right)}{\log t_1}. \tag{18}$$

For $C > 0$, this expression simplifies further at large times:

$$F(C, C; t_1) \simeq C - \frac{\log 2}{\log t_1}. \tag{19}$$

Interestingly, the leading correction does not depend on C, and it is valid only if C is not too close to 0. Figure 2 displays the plots of $F(C, C; t_1) - C$, for the same values of t_1 as in figure 1.

The last result may also be interpreted in the framework of figure 1. For a given value of C_3, the point $C(t_1)$ defined by the relation $F(C(t_1), C(t_1); t_1) = C_3$ converges to the infinite time right-angle singularity $C_1 = C_2 = C_3$ as (see equation (19)):

$$C(t_1) \simeq C_3 + \frac{\log 2}{\log t_1}. \tag{20}$$

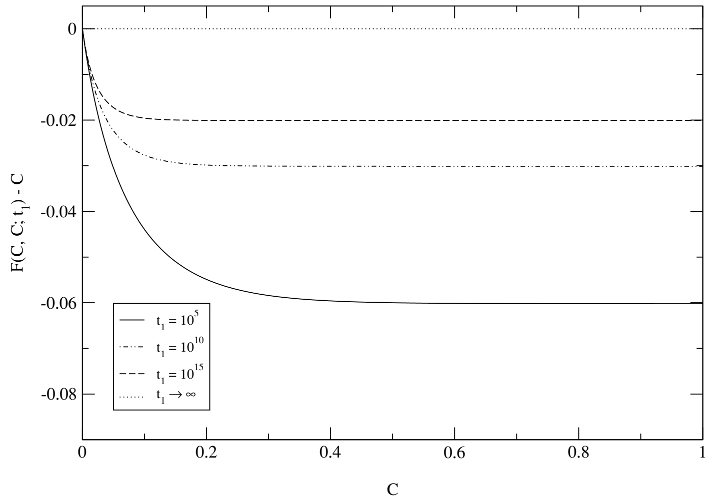

Figure 2. Plot of $F(C, C; t_1) - C$ versus C for the same values of t_1 as in figure 1. The dotted curve, corresponding to the infinite time limit $f(C, C) = C$, is added for comparison. The departure from $f(C, C) = C$ is almost independent of C, except for C close to 0.

This logarithmic correction may explain why it seems so difficult to observe the convergence towards the ultrametric relation in experimental or numerical data, where only a few decades (usually between four and six) are available.

6. The effect of 'shear'

The effect of an external 'shear' (or power injection) on ageing was investigated in the context of mean-field models in [9] and the trap model in [11]. In both models, ageing is interrupted by the shear for t_w larger than a timescale τ_r which diverges as the shear rate $\dot{\gamma}$ tends to zero. In the model considered in [11], this timescale is given by

$$\tau_r \simeq \frac{1}{\dot{\gamma}} \left(\log \frac{1}{\dot{\gamma}} \right)^{\frac{1}{2}}. \tag{21}$$

In the limit where the waiting time t_w is much larger than τ_r, the dynamics of the model becomes stationary. The (power-law) distribution of trapping times $p(\tau)$ is, in a first approximation, unaffected by the shear for $\tau \ll \tau_r$, whereas for $\tau \gg \tau_r$, $p(\tau)$ decays exponentially. So for $t \ll \tau_r$ one finds the same result as above with t_w replaced by τ_r:

$$C(t + t_w, t_w) = \mathcal{C}\left(\frac{\log t}{\log \tau_r} \right) \simeq 1 - \frac{\log t}{\log \tau_r}. \tag{22}$$

As discussed in [9, 10], dynamical ultrametricity in this context manifests itself by the appearance of an infinity of timescales in the limit $\tau_r \to \infty$ (i.e. $\dot{\gamma} \to 0$): the time needed for the correlation to decay to a certain value c diverges as τ_r^{1-c} (see [12] for a further discussion). Note that a $\log t / \log \tau_r$ scaling was already proposed in [9] in order to fit the numerical data.

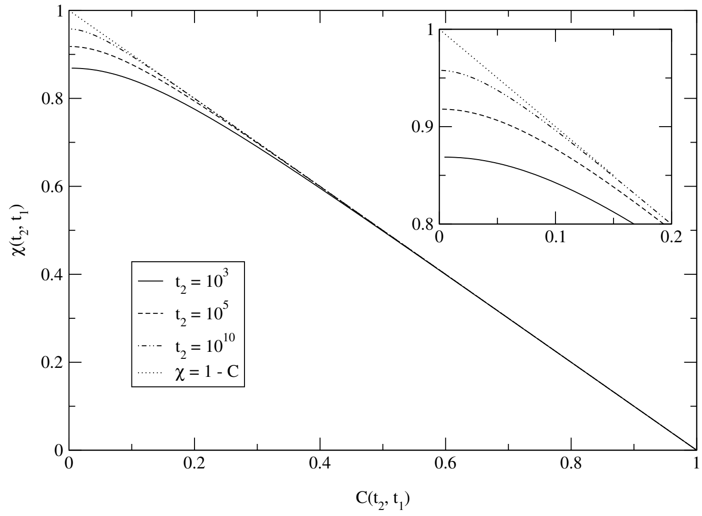

Figure 3. Plot of the integrated response $\chi(t_2, t_1)$ versus $C(t_2, t_1)$ parametrized by t_1, for $t_2 = 10^3$, 10^5 and 10^{10} from bottom to top. The local slope is equal to $-X(t_2, t_1)/T_c$, and it converges very slowly towards the asymptotic value $-1/T_c$ for small C (see inset). T_c is chosen as the temperature unit.

7. The fluctuation–dissipation ratio

It is interesting to study the fluctuation–dissipation ratio X in the trap model at the critical point[2]. This ratio is defined as

$$X(t_2, t_1) = \frac{T R(t_2, t_1)}{\partial C(t_2, t_1)/\partial t_1} \tag{23}$$

where $R(t_2, t_1)$ is the response of the system at time $t_2 > t_1$ to a small bias field applied at time t_1 (see [7, 14] for details). In the trap model, where the field only changes the trapping time of the starting site, one finds that the following relation holds in general:

$$T R(t_2, t_1) = -\frac{\partial C(t_2, t_1)}{\partial t_2}. \tag{24}$$

Using this result, one sees that in the 'liquid' phase $T > T_c$ where all two-time functions only depend on time differences, $X \equiv 1$: the usual FDT holds. In the glass phase, on the other hand, one finds that $X(t_2, t_1) = t_1/t_2$: the value of X is non-trivial in the whole scaling regime $t_2 - t_1 \sim t_1$. Right at the critical point $T = T_c$, one can express X up to first order in $1/\log t_1$ as

$$X(t_2, t_1) = \frac{\left(t_1^C - 1\right)^2}{t_1^C\left(t_1^C + 1 - C\right)} \qquad C \equiv C(t_2, t_1) > 0. \tag{25}$$

So for any fixed $C > 0$, X tends to 1 in the asymptotic limit $t_1 \to \infty$. We show in figure 3 the now famous plot of the integrated response $\chi(t_2, t_1) = \int_{t_1}^{t_2} R(t_2, t')\,\mathrm{d}t'$ versus $C(t_2, t_1)$, parametrized by $t_1 > t_2$, which should yield a straight line of slope $-1/T_c$ when the FDT holds. Again, one sees a very slow convergence towards the asymptotic result for small values of C.

[2] Note that the critical point of the trap model that we consider here is quite different from that of, say, the Ising model. The fluctuation–dissipation ratio in that case was considered in [13].

8. The multi-level trap model and discussion

The simple trap model described above can be considered as a one-step RSB model. In order to generalize the model to a full RSB one, it has been proposed in [7] (and further studied in [15]) to follow Parisi's procedure for the static solution of the SK model. Roughly speaking, it means that each trap is recursively subdivided into a new series of traps, in a hierarchical manner. Each level k of traps is characterized by a certain overlap between states q_k and by an exponential probability distribution of the energy barriers, with a critical temperature T_c^k depending on the level index k. The critical temperatures are related to Parisi's function $x_k = x(q_k)$ as $T_c^k = T/x_k$, and satisfy the relations $T_c^k < T_c^{k-1}$. At any temperature, the levels of the tree corresponding to $q > q_{\rm EA}(T)$, where $q_{\rm EA}$ is the Edwards–Anderson parameter, are such that $x_k > 1$, so that these levels are equilibrated (i.e. $T_c^k < T$).

In the single-level trap model, the correlation function $\Pi(t_w + t, t_w)$ in the ageing phase $T < T_c$ behaves at short times as

$$\Pi(t_w + t, t_w) \simeq 1 - \frac{\sin \pi x}{\pi(1-x)} \left(\frac{t}{t_w}\right)^{1-x} \qquad 1 \ll t \ll t_w \tag{26}$$

with $x = \frac{T}{T_c}$. In the multi-level model with a finite number (M) of levels, the total correlation function is defined as

$$\begin{aligned} C(t_w + t, t_w) &= \sum_{k=0}^{M} q_k [\Pi_k(t_w + t, t_w) - \Pi_{k+1}(t_w + t, t_w)] \\ &= q_0 + \sum_{k=1}^{M} (q_k - q_{k-1}) \Pi_k(t_w + t, t_w) \end{aligned} \tag{27}$$

q_k being the kth level overlap, and $\Pi_k(t_w + t, t_w)$ is the probability that the process has never jumped beyond the kth layer of the tree between t_w and $t_w + t$ with the convention that $\Pi_0(t_w + t, t_w) = 1$ and $\Pi_{M+1}(t_w + t, t_w) = 0$ (see [7] for details). From this definition, we see that $C(t_w + t, t_w)$ is dominated at short times by the levels k with x_k close to 1, for which the short time singularity is the strongest. (We assume that $T < T_c^0$, i.e. at least one level is ageing.) Therefore, we expect to observe the dynamical ultrametricity associated with the level k^* for which $x_{k^*} = 1$ in the 'short' time regime $\log t/\log t_w < 1$, before the t/t_w regime associated with the levels $k < k^*$ sets in. (Note that if $C(t_w + t, t_w)$ is a function of t/t_w, then the function $f(x, y)$ defined above cannot be equal to $\min(x, y)$.)

Interestingly, 'short time' dynamical ultrametricity exists for the hierarchical tree model in the whole low-temperature phase, but has no relation with the static ultrametricity built in the tree structure which encodes Parisi's RSB solution. Thus, the origin of dynamical ultrametricity in the generalized trap model is again very different from that found in mean-field spin-glass models, which in the latter case is deeply related to the Parisi function $x(q)$ which encodes the structure of the tree. As mentioned in the introduction, the physical interpretation of ageing in the two scenarios is radically different, although some of the phenomenology is very similar.

Ageing experiments in spin glasses have been interpreted within the framework of the multi-level trap model in [7]. The need for several levels comes not only from rejuvenation and memory in temperature shift experiments, but also from the detailed shape of the TRM relaxation at a given temperature, which shows that the short time and long time singularities are described by *different* exponents x_k [16]. The short time exponent $x \sim 0.8$ is significantly larger than the long time exponent, $x \sim 0.2$.

As discussed in [17, 18], it is natural to interpret the different levels of the hierarchy in terms of length scales, associating a value of $x = x(\ell)$ with each length scale ℓ, such that

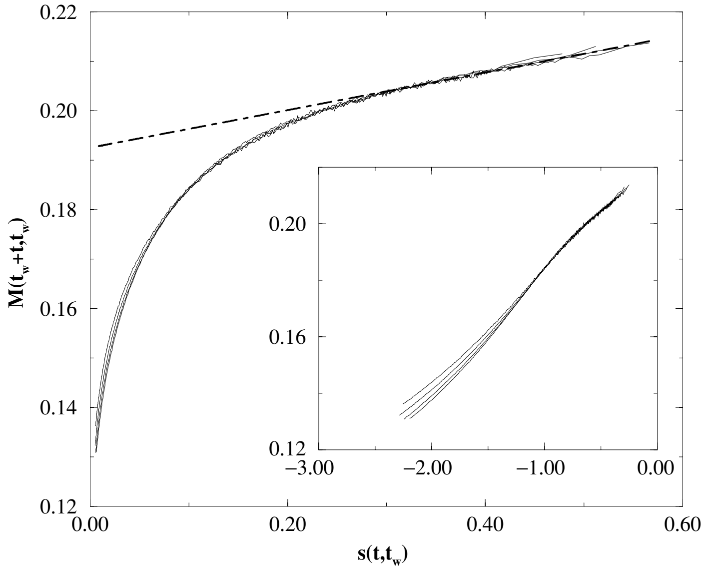

Figure 4. Plot of the TRM data in a AgMn spin glass at $T/T_g = 0.75$, for $t_w = 300$, 1000, 3000 and 10 000 s. The horizontal axis is the variable $s(t, t_w)$ defined in the text. The rescaling is very good for $s(t, t_w) > 0.3$ approximately (or $\log t/\log t_w < 0.7$), but becomes inadequate for longer times t, as more clearly seen in the inset where $\log s$ is used. The dashed line is an affine fit of the short time part of the data.

$x(\ell) \propto \ell^{-\theta}$, where $\theta > 0$ is the 'energy' exponent. Therefore, one can expect that for any temperature within the spin-glass phase, there will be a particular 'critical' length scale ℓ^* such that $x(\ell^*) = 1$. This particular length scale will therefore contribute to the correlation and response as a function of $\log t/\log t_w$.

We have re-analysed some TRM data in the spirit of the present discussion, assuming that FDT holds in the regime $t \ll t_w$. We show in figure 4 the decay of the TRM in AgMn at $T = 0.75T_g$ [19], plotted as a function of $s(t, t_w) = \log(1 + t_w/t)/\log t_w$, as suggested by equation (2). This figure shows that the rescaling is very good at short times, but is violated for large t/t_w (corresponding to small $s(t, t_w)$). From equation (27), we indeed expect to observe the sum of a contribution $\mathcal{M}(k < k^*)$ from the levels $k < k^*$ (that only depends on t/t_w) and a contribution from $k \simeq k^*$, proportional to $s(t, t_w)$, and finally a non-ageing contribution from $k > k^*$ that can be assumed to be small. When $\alpha = \log t/\log t_w < 1$, the first contribution does not vary much, since $(t/t_w)^{1-x_k} = t_w^{(\alpha-1)(1-x_k)}$ is very small for large t_w, whereas the second contribution is a function of α. This suggests that in the short time regime one should observe

$$M(t_w + t, t_w) = \mathcal{M}(k < k^*) + m_1 s(t, t_w) \tag{28}$$

where m_1 is a certain prefactor. The quantity $\varphi = m_1/(\mathcal{M}(k < k^*) + m_1)$ measures the relative contribution of the levels $k < k^*$ and $k \simeq k^*$ to the total decay of the signal. The dashed line shown in figure 4 is a linear fit of the initial decay, as a function of s, from which one extracts (in this particular example) $\varphi \approx 0.2$.

In all the above formulae, time is implicitly measured in units of a microscopic time τ_0. The value of τ_0 is not necessarily an individual flip time $\sim 10^{-12}$ s, since collective dynamics may exist in the vicinity of the transition point (see [20, 21] for a detailed discussion). In figure 4, we have chosen $\tau_0 = 10^{-5}$ s to achieve the best rescaling. This value is very close to that extracted from the analysis of [20].

The conclusion is that the TRM data are indeed compatible with a $\log t/\log t_w$ behaviour for short times. Note that this behaviour is tantamount to a logarithmic dependence of the ac susceptibility or to $1/f$ noise. We have insisted in previous papers [7, 15] on the fact that the existence of levels in the vicinity of $x = 1$ generically leads to $1/f$ noise for long times and low frequencies, because the contribution of other levels (both above and below k^*) becomes negligible in that regime. Here we show that the degrees of freedom contributing to $1/f$ noise also give rise to dynamical ultrametricity.

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Appendix

Here we report the detailed calculation of the correlation function $\Pi(t_w + t, t_w)$ for $T = T_c$. Let us define the Laplace transform of $\Pi(t_w + t, t_w)$ with respect to t_w:

$$\hat{\Pi}(t, E) = \int_0^\infty \mathrm{d}t_w\, \mathrm{e}^{-Et_w}\, \Pi(t_w + t, t_w). \tag{29}$$

It was shown in [7] (using a slightly different definition of the Laplace transform) that

$$\hat{\Pi}(t, E) = \frac{1}{E} \frac{\left\langle \frac{\tau}{E\tau+1}\, \mathrm{e}^{-t/\tau} \right\rangle}{\left\langle \frac{\tau}{E\tau+1} \right\rangle} \tag{30}$$

where $\langle \cdots \rangle$ denotes the average over the distribution of trapping times $\rho(\tau)$. Taking τ_0 as the time unit, the denominator reads

$$\left\langle \frac{\tau}{E\tau + 1} \right\rangle = \int_1^\infty \frac{\mathrm{d}\tau}{\tau^2} \frac{\tau}{E\tau + 1} \tag{31}$$

$$= \int_E^\infty \frac{\mathrm{d}u}{u(1+u)} \tag{32}$$

$$= \log\left(1 + \frac{1}{E}\right). \tag{33}$$

Defining $\hat{f}(E)$ as

$$\hat{f}(E) \equiv \frac{1}{E \log\left(1 + \frac{1}{E}\right)} \tag{34}$$

we can write $\hat{\Pi}(t, E)$ as

$$\hat{\Pi}(t, E) = \hat{f}(E) \int_1^\infty \frac{\mathrm{d}\tau}{\tau^2} \frac{\tau}{E\tau + 1}\, \mathrm{e}^{-t/\tau}. \tag{35}$$

Using the relation

$$\frac{\tau}{E\tau + 1} = \int_0^\infty \mathrm{d}t_w\, \mathrm{e}^{-Et_w}\, \mathrm{e}^{-t_w/\tau} \tag{36}$$

we get

$$\hat{\Pi}(t, E) = \hat{f}(E) \int_0^\infty \mathrm{d}t_w\, \mathrm{e}^{-Et_w} \int_1^\infty \frac{\mathrm{d}\tau}{\tau^2}\, \mathrm{e}^{-(t_w+t)/\tau} \tag{37}$$

$$= \hat{f}(E) \int_0^\infty \mathrm{d}t_w\, \mathrm{e}^{-Et_w} \frac{1 - \mathrm{e}^{-(t_w+t)}}{t_w + t}. \tag{38}$$

Performing the inverse Laplace transform, we obtain a convolution:

$$\Pi(t_w + t, t_w) = \int_0^{t_w} \mathrm{d}u\, f(u) \frac{1 - \mathrm{e}^{-(t+t_w-u)}}{t + t_w - u} \tag{39}$$

where $f(u)$ is the inverse Laplace transform of $\hat{f}(E)$. The asymptotic large u behaviour of $f(u)$ is readily calculated using a tauberian theorem [22]

$$f(u) = \frac{1}{\log u} + \mathcal{O}\left(\frac{1}{(\log u)^2}\right) \qquad u \to \infty. \tag{40}$$

Assuming now that $t \gg 1$, one can neglect the exponential term in (39), to find

$$\Pi(t_w + t, t_w) \simeq \int_0^{t_w} \mathrm{d}u \frac{f(u)}{t + t_w - u}. \tag{41}$$

We wish to calculate $\Pi(t_w + t, t_w)$ for very large t_w, up to first order in $\frac{1}{\log t_w}$. Since we only know the large u behaviour of $f(u)$, it is convenient to separate the above integral into two parts Π_1 and Π_2, introducing a bound A such that $1 \ll A \ll t_w$. The first term reads

$$\Pi_1 = \int_0^A \mathrm{d}u \frac{f(u)}{t + t_w - u} \leqslant \frac{1}{t + t_w - A} \int_0^A f(u)\, \mathrm{d}u \tag{42}$$

so that this term should be of order $\frac{A}{t_w}$ with logarithmic corrections, as can be deduced from the small E behaviour of $\hat{f}(E)$. Now, making a change of variable $u = t_w\, \mathrm{e}^{-v \log t_w}$ in the second term Π_2, and keeping only the first term in the expansion of $f(u)$:

$$\Pi_2 = \int_0^{1-\frac{\log A}{\log t_w}} \mathrm{d}v \frac{\mathrm{e}^{-v \log t_w}}{1 + \frac{t}{t_w} - \mathrm{e}^{-v \log t_w}} \frac{1}{1 - v}. \tag{43}$$

For large $\log t_w$, the integral is dominated by the vicinity of $v = 0$, so that a small v expansion of the last factor $1/(1 - v)$ can be performed:

$$\Pi_2 = \int_0^{1-\frac{\log A}{\log t_w}} \mathrm{d}v \frac{\mathrm{e}^{-v \log t_w}}{1 + \frac{t}{t_w} - \mathrm{e}^{-v \log t_w}} (1 + v + \mathcal{O}(v^2)). \tag{44}$$

Replacement of the upper bound by ∞ yields corrections of order $1/t_w$, which can be neglected. The nth term of the v expansion leads to a contribution of order $\left(\frac{1}{\log t_w}\right)^{n+1}$, as can be seen using the new variable $z = u \log t_w$. So one can keep only the first term, which reads

$$\int_0^\infty \mathrm{d}v \frac{\mathrm{e}^{-v \log t_w}}{1 + \frac{t}{t_w} - \mathrm{e}^{-v \log t_w}} = \frac{1}{\log t_w} \int_0^1 \frac{\mathrm{d}x}{1 + \frac{t}{t_w} - x} \tag{45}$$

$$= \frac{1}{\log t_w} \log\left(1 + \frac{t_w}{t}\right). \tag{46}$$

Moreover, the second term in the large u expansion of $f(u)$ also leads to a $(\log t_w)^{-2}$ correction. As a conclusion, it has been shown that the correlation function $\Pi(t_w + t, t_w)$, in the limit of large t_w and t, is given by

$$\Pi(t_w + t, t_w) = \frac{\log\left(1 + \frac{t_w}{t}\right)}{\log t_w} + \mathcal{O}\left(\frac{1}{(\log t_w)^2}\right) \tag{47}$$

where the coefficient in front of the subdominant term could be computed exactly, if needed.

References

[1] Cugliandolo L F and Kurchan J 1994 *J. Phys. A: Math. Gen.* **27** 5749
[2] Bouchaud J-P, Cugliandolo L, Kurchan J and Mézard M 1998 *Spin-Glasses and Random Fields* ed A P Young (Singapore: World Scientific) and references therein
[3] Ben Arous G, Bovier A and Gayrard V 2001 *Preprint* cond-mat/0110223
[4] Bouchaud J-P 1992 *J. Physique I* **2** 1705
[5] Monthus C and Bouchaud J-P 1996 *J. Phys. A: Math. Gen.* **29** 3847
[6] Rinn B, Maass P and Bouchaud J-P 2001 *Phys. Rev.* B **64** 104417
[7] Bouchaud J-P and Dean D S 1995 *J. Physique* I **5** 265
[8] Dornic I, Private communication
Dornic I 1998 *PhD Thesis* Université de Nice Sophia-Antipolis and CEA Saclay
[9] Berthier L, Barrat J L and Kurchan J 2000 *Phys. Rev.* E **63** 016105
[10] Stariolo D A 2001 *Europhys. Lett.* **55** 726
[11] Sollich P, Lequeux F, Hebraud P and Cates M 1997 *Phys. Rev. Lett.* **70** 2020
Sollich P 1998 *Phys. Rev.* E **58** 738
Fielding S M, Sollich P and Cates M 2000 *J. Rheol.* **44** 323
[12] Kurchan J 2001 *Preprint* cond-mat/0110628
[13] Godrèche C and Luck J M 2000 *J. Phys. A: Math. Gen.* **33** 9141
Godrèche C and Luck J M 2000 *J. Phys. A: Math. Gen.* **33** 1151
[14] Sollich P, Fielding S and Mayer P 2001 *Preprint* cond-mat/0111241
[15] Sasaki M and Nemoto K 2000 *J. Proc. Soc. Jpn.* **69** 2283
[16] Vincent E, Hammann J, Ocio M, Bouchaud J-P and Cugliandolo L F 1996 *Proc. Sitges Conf. on Glassy Systems* ed E Rubi (Berlin: Springer)
[17] Balents L, Bouchaud J-P and Mézard M 1996 *J. Physique* I **6** 1007
[18] Bouchaud J-P 2000 *Soft and Fragile Matter* ed M E Cates and M R Evans (Bristol: Institute of Physics Publishing)
[19] Alba M, Ocio M and Hammann J 1986 *Europhys. Lett.* **2** 45
Alba M, Ocio M and Hammann J 1985 *J. Phys. Lett.* **46** L-1101
Alba M, Hammann J, Ocio M and Refregier Ph 1987 *J. Appl. Phys.* **61** 3683
Vincent E, Hammann J and Ocio M 1992 *Recent Progress in Random Magnets* ed D H Ryan (Singapore: World Scientific) p 207
[20] Dupuis V, Vincent E, Bouchaud J-P, Hammann J, Ito A and Aruga Katori H 2001 *Phys. Rev.* B **64** 174204
[21] Bouchaud J-P, Dupuis V, Hammann J and Vincent E 2002 *Phys. Rev.* B **65** 024439
[22] Feller W 1966 *An Introduction to Probability Theory and its Applications* vol II (New York: Wiley) ch XIII